Circular Motion

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ANGULAR VELOCITY (INSTANTANEOUS)

If a particle covers a small angle $= d\theta$ in a small time interval dt, then its angular velocity is defined as

$$\boxed{\omega = \frac{d\theta}{dt}}$$

Parallel to average angular velocity, we can write average angular velocity as

$$\boxed{V_{avg.} = \frac{\Delta\theta}{\Delta t}}$$

ANGULAR ACCELERATION

If the angular velocity of a particle changes by $d\omega$ in a small time interval dt, then the angular acceleration is defined as

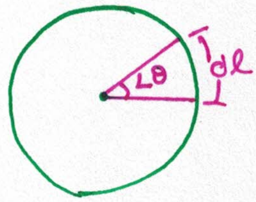
$$\boxed{\alpha = \frac{d\omega}{dt}}$$

also

$$\alpha = \omega . \frac{d\omega}{d\theta}$$

Analogy between linear Kinematics & Circular Kinematics:

(Linear Kinematics) Differential Equation	Valid for constant acceleration		
	Circular Kinematics Differential eq^n.	Linear Kinematics Solution	Circular Angular Kinematics Solution
1.) $v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$	$S = ut + \frac{1}{2}at^2$	$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
2.) $a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$	$v = u + at$	$\omega = \omega_0 + \alpha t$
3.) $a = v\frac{dv}{ds}$	$\alpha = \omega\frac{d\omega}{d\theta}$	$v^2 - u^2 = 2as$	$\omega^2 - \omega_0^2 = 2\alpha\Delta\theta$
		$r = r_0 + ut + \frac{1}{2}at^2$	$\theta = \theta + \omega_0 t + \frac{1}{2}\alpha t^2$



$$v = \frac{dl}{dt}$$

$$\omega = \frac{d\theta}{dt}$$

$$dl = R.d\theta$$

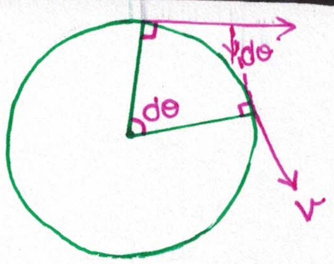
$$v = R\frac{d\theta}{dt}$$

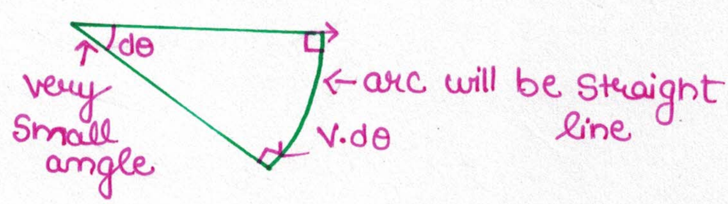
$$\boxed{v = R.\omega}$$

UNIFORM CIRCULAR MOTION

When a particle moves along a circle with constant speed, we call it a uniform circular motion.

CENTRIPITAL ACCELERATION





$$|\Delta\vec{v}| = v.d\theta$$

$$|d\vec{v}| = v.d\theta$$

$$a_c = \frac{|d\vec{v}|}{dt} = \frac{v\,d\theta}{dt}$$

$$a_c = v\omega = \frac{v^2}{R} = \omega^2 R \qquad (v = R\omega)$$

Q.) The seconds hand of a clock is ½ m long. Find
(a) Its angular velocity
(b) The speed of the tip
(c) Angular acceleration of the tip
(d) Centripital acceleration of the tip.

(a) $\omega = \frac{2\pi}{60} = \frac{\pi}{30}$ rad/sec

(b) $v = R\times\omega$

$= \frac{1}{2}\times\frac{\pi}{30} = \frac{\pi}{60}$ m/s

(c) $\alpha = 0$ (uniform circular motion)

(d) $a_c = v\omega = \frac{\pi}{60} \times \frac{\pi}{30} = \frac{\pi^2}{1800}\ m/s^2$

NON-UNIFORM CIRCULAR MOTION

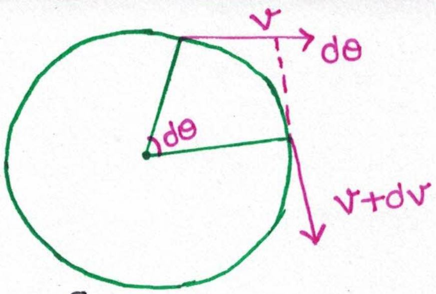


Fig.(1)

$$a = \frac{d\vec{v}}{t}$$

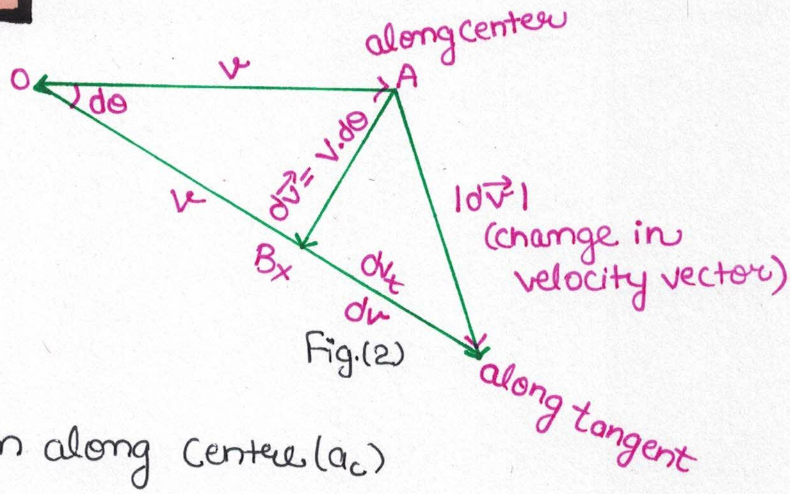


Fig.(2)

Component of acceleration along centre (a_c)

$$a_c = \frac{dV_c}{dt}$$

$$= \frac{v\,d\theta}{dt}$$

$$a_c = v\omega = \frac{V^2}{R} = \omega^2 R$$

Component of acceleration along tangent (a_t)

$$a_t = \frac{dv_t}{dt} = \frac{dv}{dt} = \frac{d(R\omega)}{dt} = R\alpha$$

$$\therefore \vec{a} = \vec{a_c} + \vec{a_t}$$

$$\vec{a} = v\frac{d\theta}{dt}\hat{c} + \frac{dv}{dt}\hat{t}$$

Consider a particle with initial speed v. Lets its speed after time dt be (v+dv). Fig (1) & (2)

We can resolve the change in velocity vector into two components AB & BD.

The line AB is drawn by putting compass point at O and radius equal to OA.

Thus,
$$OB = v$$
$$BD = |dv|$$
$$AB = v.d\theta$$

Therefore we can write change in velocity as $\vec{AB}\,\hat{c} + \vec{CD}\,\hat{t}$, where $\hat{c}$ & $\hat{t}$ represents the unit vectors along tangent and center.

$$\boxed{\frac{d\vec{v}}{dt} = \frac{v\,d\theta}{dt}\hat{c} + \frac{dv}{dt}\hat{t}}$$

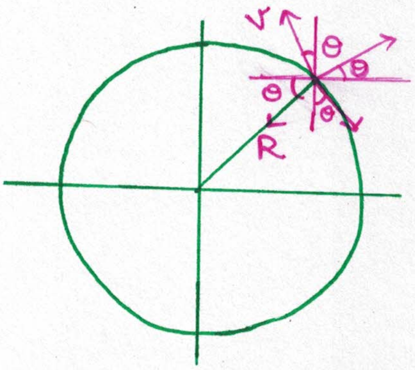
NOTE: Magnitude of total acceleration is given by $\sqrt{a_t^2 + a_c^2}$

$$\hat{c} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j}$$

$$\hat{t} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\vec{v} = v\hat{t}$$

$$\frac{d\vec{v}}{dt} = \frac{v\,d\hat{t}}{dt} + \hat{t}\frac{dv}{dt}$$



$$\frac{d\hat{t}}{dt} = -\cos\theta.\frac{d\theta}{dt}\hat{i} - \sin\theta.\frac{d\theta}{dt}\hat{j}$$

$$= \omega\underbrace{(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j})}_{\hat{c}}$$

$$\frac{d\vec{v}}{dt} = v\omega\,\hat{c} + \frac{dv}{dt}.\hat{t}$$

Q.) A particle moves in a circle of radius R such that its speed is given by $v = t$

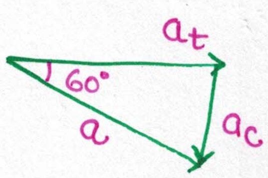
(i) Find its tangential acceleration.

(ii) What is its centripital acceleration at a general time.

(iii) After how much time does the total acceleration make an angle of 60° with tangential acceleration.

$$\text{(i)}\quad a_T = \frac{dv}{dt} = \frac{dt}{dt} = 1$$

$$\text{(ii)}\quad a_c = \frac{v^2}{R} = \frac{t^2}{R}$$



$$\text{(iii)}\quad \tan 60° = \frac{a_c}{a_t}$$

$$\sqrt{3} = \frac{t^2}{R \times 1}$$

$$t = \sqrt{R\sqrt{3}}\ \text{sec.}$$

Q.) The speed of a particle moving along a circle of Radius R is given by $v = \sqrt{t}$. Find

(i) The angle covered by a particle at a general time

(ii) angular acceleration of the particle

(iii) The time at which tangential acceleration are equal.

$$\text{(i)}\quad R \times \omega = \sqrt{t}$$

$$\frac{d\theta}{dt} = \frac{\sqrt{t}}{R}$$

$$\int_0^\theta d\theta = \int_0^t \frac{\sqrt{t}}{R} \times dt$$

$$\theta = \frac{2}{3} Rt\sqrt{t}$$

(ii) $\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right)$

$$\alpha = \frac{d}{dt}\left(\frac{\sqrt{t}}{R}\right) = \frac{d}{dt}\left(\frac{t^{1/2}}{R}\right)$$

$$\alpha = \frac{1}{2R\sqrt{t}}$$

(iii) $a_c = \frac{v^2}{R} = \frac{t}{R}$

$$a_t = R\alpha = \frac{1}{2\sqrt{t}}$$

$$\frac{t}{R} = \frac{1}{2\sqrt{t}} \Rightarrow (t)^{3/2} = \frac{R}{2}$$

$$t = \left(\frac{R}{2}\right)^{2/3}$$

GENERAL PLANE MOTION AS A SERIES OF INFINITESIMAL CIRCULAR ARCS

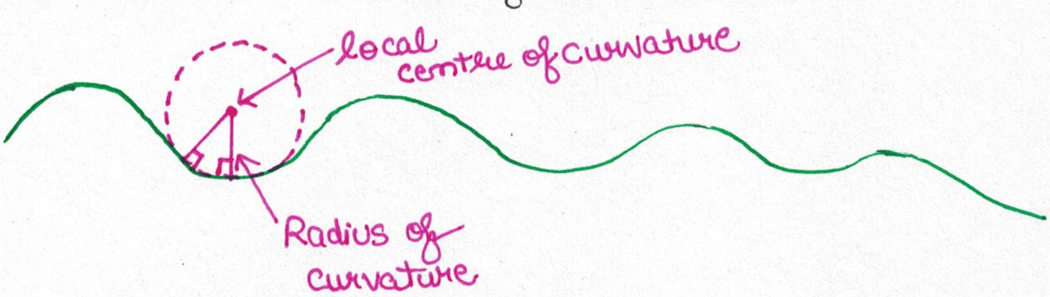
Any motion along a general curve in a plane can be thought as a series of infinitesimal arcs with centre of curvature and radius of curvature defined locally as follows:

LOCAL CENTRE OF CURVATURE

Draw the perpendiculars to the curve from two neighbouring points. The point of intersection of these perpendiculars is called local centre of curvature at those neighbouring points.

LOCAL RADIUS OF CURVATURE

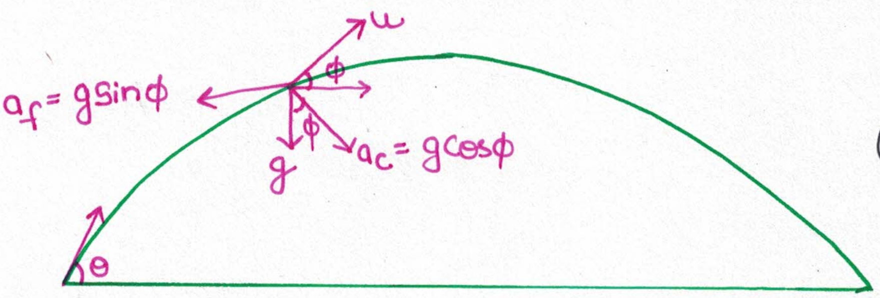
Distance between neighbouring points to the centre of curvature is called the local radius of curvature.



NOTE: Since the small segment of a general curve is that like a circular arc, we can apply the concept of circular motion to a general curve as well.

i.e.
$$a_t = \frac{dv}{dt} = R\alpha$$
$$a_c = \frac{v^2}{R} = \omega^2 R = v\omega$$
$$R = \frac{v^2}{a_c}$$
$$a = \sqrt{a_c^2 + a_t^2}$$

Q) Find the radius of curvature of a projectile launched with speed u at angle θ when it is flying at an angle ϕ with the horizontal. Where is the radius of curvature maximum and where is it minimum?



$$\left(v = \frac{u\cos\theta}{\cos\phi}\right)$$

$$R = \frac{v^2}{a_c} = \frac{u^2\cos^2\theta}{\cos^2\phi}\cdot\frac{1}{g\cos\phi}$$

$$\boxed{R = \frac{u^2\cos^2\theta}{g\cos^3\phi}}$$

For $R_{min.}$ $\cos\phi = 1$ or $\phi = 0°$ (highest point)

For $R_{max.}$, $\cos\phi$ should be minimum or ϕ should be max.
$$\phi = \theta$$
$$\therefore \frac{u^2}{g\cos\theta} = R$$